Some Width Parameters and Even-Hole-Free Graphs

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1. Even-hole-free graphs

- H is an induced subgraph of G if H can be obtained from G by *deleting vertices* (denoted by $H \subseteq_{\text{ind}} G$)

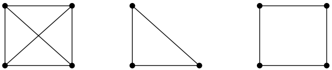


Figure: A graph, an induced subgraph, and a non-induced subgraph

- G is H-free if no induced subgraph of G is isomorphic to H
- When $\mathcal{F}$ is a family of graphs, $\mathcal{F}$-free means H-free, $\forall H \in \mathcal{F}$

So, a graph is even-hole-free (EHF) if it *does not contain* even holes as induced subgraph.

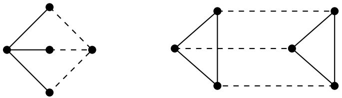


Figure: Even-hole-free graphs are (theta, prism)-free

2. Tree-width ($tw(G)$)

- Tree-width is a graph parameter (integer ≥ 1) that describes the structural complexity of the graph.
- It measures how close G from being a tree.
- This notion is introduced in the graph-minor-theory papers of Robertson & Seymour. This was initially defined by Halin (1976).

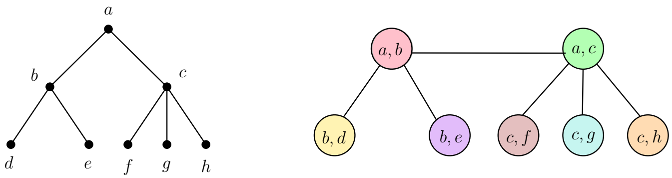
2a. Computing tree-width trough tree decomposition

Tree decomposition of G is a pair $\left(T, (B_x \subseteq V(G))_{x \in V(T)}\right)$:

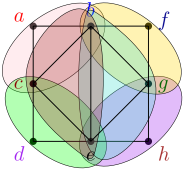
- T is a tree
- $\{B_x\}_{x \in V(T)}$ is a collection of bags.

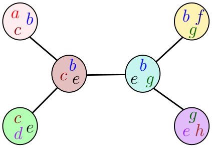
such that:

- $\forall v \in V(G)$, $\exists x \in V(T)$ s.t. $v \in B_x$;
- $\forall v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty subtree of T;
- $\forall vw \in E(G)$, there is some bag B_x containing both v and w.



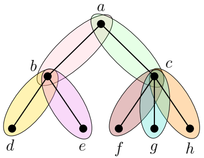
2a. Computing tree-width trough tree decomposition

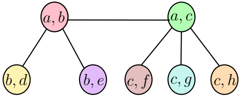


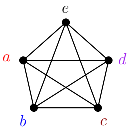


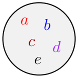
AXIOMS

1. Every vertex is in a bag
2. Every edge is in a bag
3. $\forall v \in V(G)$, the support of v forms a subtree









- width of T is the size of the largest bag - 1
- tree-width of G is the width of the optimal tree decomposition of G

2b. Computing tree-width trough chordalization

$$\text{tw}(G) = \min_{H \text{ chordalization of } G} \{\omega(H) - 1\}$$

- Chordal graphs are graphs possessing no hole (chordless cycle)
- A chordalization of G is a graph H obtained by adding edges to G, such that H is chordal

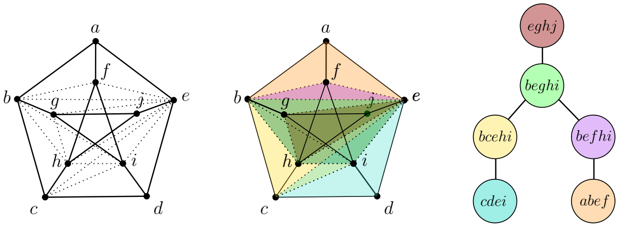


Figure: A chordalization of a graph and its tree-like structure

2c. Computing tree-width trough bramble

- A bramble for a graph G is a family of connected subgraphs of G that all touch each other: for every X and Y in the bramble, either X and Y share a vertex or an edge.

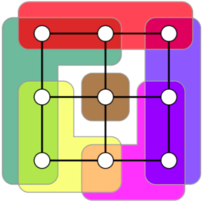


Figure: A bramble of *order 4* in a 3×3 grid graph, consisting of six mutually touching connected subgraphs (source: wikipedia)

- The order of a bramble is the smallest size of a hitting set, a set of vertices of G that has a nonempty intersection with each of the subgraphs.

2d. Complexity of computing tree-width

- Determining whether a given graph G has tree-width at most a given variable k is NPC [Arnborg et al. (1987)].
- If k is fixed, the graphs with tree-width k can be recognized, and constructing a tree decomposition of width k is in $\mathcal{O}(1)$ [Bodlaender (1996)].

Theorem (Courcelle (1990))

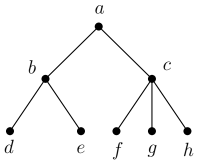
Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded tree-width.

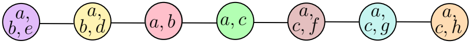
$\rightarrow$ by dynamic programming using the tree decomp. of the graphs.
$\rightarrow$ Graph problems expressible in MSO: coloring, MIS, etc.

3. Path-width ($pw(G)$)

Path decomposition is a special type of tree decomposition. Hence,

$$pw(G) \geq tw(G)$$





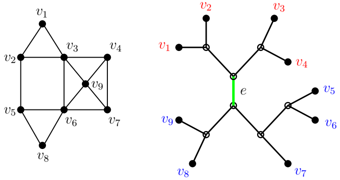
Theorem (Cygan, et al. (2015))

Let G be a graph, and I be an interval graph that contains G as a subgraph (possibly not induced). Then $pw(G) \leq \omega(I) - 1$, where $\omega(I)$ is the size of the maximum clique of I.

Interval graph: intersection graphs of a set of subpaths of a path.

4. Rank-width ($rw(G)$)

$rw(G) = k \in \mathbb{Z}^+$ if G can be decomposed into tree-like structures by splitting $V(G)$ s.t. each cut induces a matrix of rank $\leq k$.



$$\text{width}(e) = \text{rank} \begin{pmatrix} & v_1 & v_2 & v_3 & v_4 \\ v_5 & 0 & 1 & 0 & 0 \\ v_6 & 0 & 0 & 1 & 0 \\ v_7 & 0 & 0 & 0 & 1 \\ v_8 & 0 & 0 & 0 & 0 \\ v_9 & 0 & 0 & 1 & 1 \end{pmatrix} = 3$$

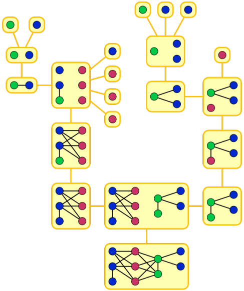
Hlineny et. al. Width parameters beyond tree-width and their applications, The Computer Journal (51), 2008

Rank decomposition is a cubic tree $\mathcal{T}$, with a bijection $\nu : V(G) \to \mathcal{L}(\mathcal{T})$

- width(e) : cut-rank of the adjacency matrix of the separation
- width($\mathcal{T}$) : $\max\{\text{width}(e) \mid e \in E(\mathcal{T})\}$
- rank-width of G is the width of the "best" rank decomposition

5. Clique-width ($cw(G)$)

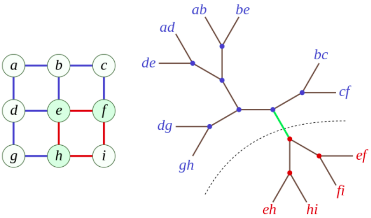
$cw(G)$ is *the minimum number of labels* needed to construct G by a sequence of the following operations: *disjoint unions*, *relabelings*, and *label-joins*.



1. Creation of a new vertex v with label i (noted $i(v)$)
2. Disjoint union of two labeled graphs G and H (denoted $G \oplus H$)
3. Joining by an edge every vertex labeled i to every vertex labeled j (denoted $\nu(i,j)$), where $i \neq j$
4. Renaming label i to label j (denoted $\rho(i,j)$)

Figure: Construction of a distance-hereditary graph of clique-width 3 (source: wikipedia)

6. Branch-width ($bw(G)$)



- e-partition is the partition of T into subtrees T_1 and T_2 by cutting T on the edge e.
- The width of an e-separation is the number of vertices of G that are incident both to an edge of E_1 and to an edge of E_2;
- The branchwidth of G is the minimum width of any branch-decomposition of G.

7. Relation between width parameters

Lemma (Corneil, Rotics (2005) and Oum, Seymour (2006))

For every graph G, the followings hold:

- $\text{rw}(G) \leq \text{cw}(G) \leq 2^{\text{rw}(G)+1}$;
- $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G)-1}$;
- $\text{tw}(G) \leq \text{pw}(G)$.

Notation: rw: *rank-width*, cw: *clique-width*, tw: *tree-width*, pw: *path-width*

7. Relation between width parameters

- Graph classes of *bounded tree-width* are necessarily *sparse*.
- There exist *dense* graph classes with *bounded clique-width*.

Clique-width of K_n is 2, but the tree-width is $n - 1$.

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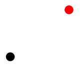

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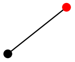

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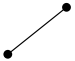

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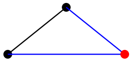

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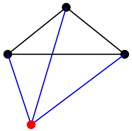

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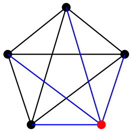

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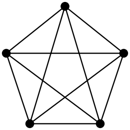

Clique-width of K_n is 2, but the tree-width is $n-1$.

8. Results on width of EHF graphs

a. Planar EHF $\rightarrow tw \leq 49$ [Silva, da Silva, Sales (2010)]

b. Pan-free EHF $\rightarrow tw \leq 1.5\omega(G) - 1$ [Cameron, Chaplick, Hoàng (2015)]

c. K_3-free EHF $\rightarrow tw \leq 5$ [Cameron, da Silva, Huang, Vušković (2018)]

d. Cap-free EHF $\rightarrow tw \leq 6\omega(G) - 1$ [Cameron, da Silva, Huang, Vušković (2018)]

e. EHF without star cutset $\rightarrow$ bounded rank-wd [Le (2018)]

f. Diamond-free EHF $\rightarrow$ unbounded rank-wd [Adler et al. (2018)]

g. K_4-free EHF $\rightarrow$ unbounded tree-wd [S., Trotignon (2019)]

h. EHF without K_k-minor $\rightarrow tw \leq f(k)$ [Aboulker, et al. (2020)]

i. EHF with maximum degree $\leq 3 \rightarrow tw \leq 3$ [Aboulker, et al. (2020)]

j. EHF with bounded maximum degree (i.e. any $\Delta(G) = d$) $\rightarrow tw \leq f(d)$ [Abrishami, Chudnosky, Vušković (2021)]

8a. Tree-width and grid-minor

- Planar EHF graphs have tree-width ≤ 49.

H is a minor of G if H can be formed from G by deleting edges and vertices and by contracting edges.

Theorem
If H is a minor of G, then $tw(H) \leq tw(G)$.

Let $G_{(r \times r)}$ be the the largest square grid-minor in G,

- Since $tw(G_{(r \times r)}) = r$, we have $tw(G) \geq r$.
- The grid-minor-theorem (Robertson & Seymour): $\exists f$ a function s.t. $tw(G) \leq f(r)$

8a. Tree-width and grid-minor

- If G is planar and does not contain a $(k \times k)$-grid as a minor, then $tw(G) \leq 6k - 5$ [Robertson, Seymour, Thomas (1994)].

Theorem

Every planar even-hole-free graph has tree-width at most 49 [Silva, da Silva, Sales (2010)].

- Any $G_{(9\times 9)}$-model of minor contains a *theta* or a *prism* (which contains an even hole).

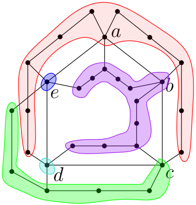

Figure: An example of a *model* of K_5-minor

8b. Results on width of EHF graphs

Theorem (Cameron, Chaplick, Hoàng (2015))

Every (even hole, pan)-free graph G satisfies $tw(G) \leq 1.5\omega(G) - 1$

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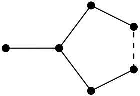


Proof. skipped

8c. EHF triangle-free graphs have bounded *tw* and *cw*

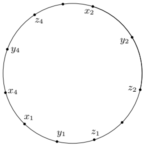
Let $\mathcal{C}$ be the class of (triangle, theta, even wheel)-free graphs.

Theorem

Every $G \in \mathcal{C}$ satisfies $tw(G) \leq 5$ [Cameron, da Silva, Huang, Vušković (2018)].

Theorem (Conforti, Cornuéjols, Kapoor, Vušković (2000))

Every $G \in \mathcal{C}$ either (i) contains at most three vertices; (ii) is the cube; (iii) has no K_1 or K_2 separator; or it can be obtained, starting from a hole, by a sequence of good ear additions.



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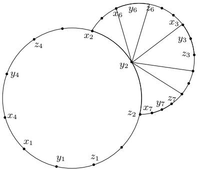

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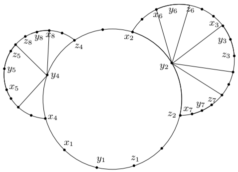

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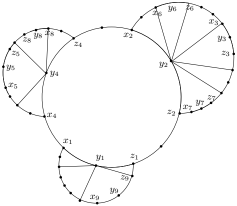

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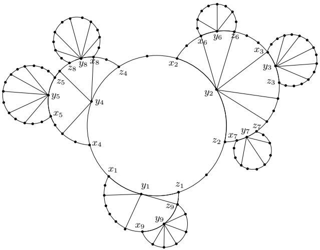

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Chordalization technique applied to the constructed graph.

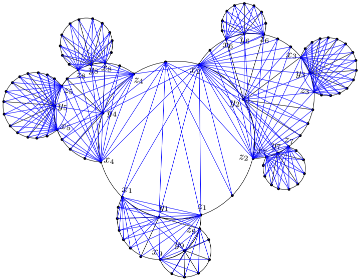

Figure: Every graph built in this way is a subgraph of a chordal graph with $\omega = 6$

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Recall that Corneil and Rotics (2005) show that

$$cw(G) \leq 3 \times 2^{tw(G)-1}$$

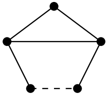
Corollary

Every (even hole, triangle)-free graph G satisfies $cw(G) \leq 48$.

8d. EHF cap-free graphs have ω-bounded *tw*

Theorem (Cameron, da Silva, Huang, Vušković (2018))

An (even hole, cap)-free graph G satisfies $tw(G) \leq 6\omega(G) - 1$.



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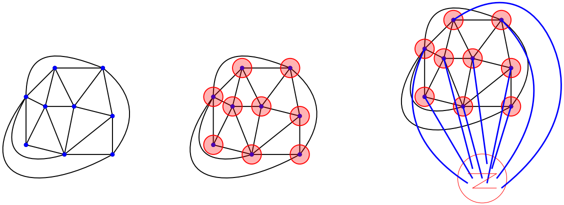
An (even hole, cap)-free graph G satisfies $tw(G) \leq 6\omega(G) - 1$.

Theorem (Cameron, da Silva, Huang, Vušković (2018))

Every (even-hole, cap)-free graph G is obtained from a maximal induced subgraph F of G with at least 3 vertices, by first blowing up vertices of F into cliques, and then adding a universal clique.



$\omega_\Delta = 6$

$\omega_\Delta = \max_v 6|K_v| \leq 6(\omega(G) - |U|)$

$\omega_\Delta \leq 6(\omega(G) - |U|) + |U| \leq 6\omega(G)$

8e. EHF graphs having no star cutset

Theorem (Le (2018))

Every even-hole-free graph G with no star cutset has rank-width at most 3.

- Every $G \in \mathcal{C}$ can be decomposed using 2-join.

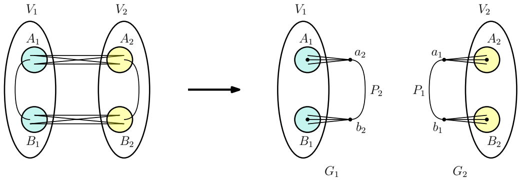


Figure: Scheme of a 2-join decomposition

- The set of basic graphs: cliques, holes, long pyramid, extended nontrivial basic graphs.

8e. EHF graphs having no star cutset

Theorem (Le (2018))

Every even-hole-free graph G with no star cutset has rank-width at most 3.

- The set of basic graphs: cliques, holes, long pyramid, extended nontrivial basic graphs.
- Rank-decomposition of the merged graph in $\mathcal{C}$

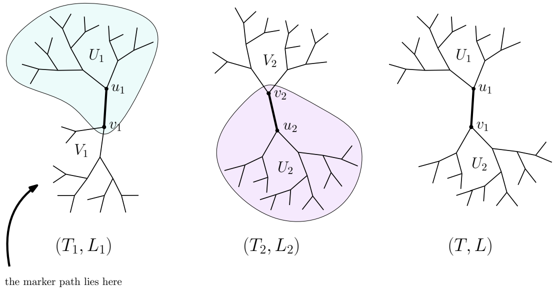


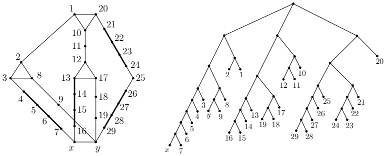
Figure: Rank-decomposition of the two blocks G_1 and G_2, and a rank-decomposition of G obtained by identifying $u_1 v_1$ and $v_2 u_2$

8e. EHF graphs having no star cutset

Theorem (Le (2018))

Every even-hole-free graph G with no star cutset has rank-width at most 3.

- The set of basic graphs: cliques, holes, long pyramid, extended nontrivial basic graphs.
- Every basic graph has rank-width at most 3: $rw(\text{clique}) \leq 1$, $rw(\text{hole}) \leq 2$, $rw(\text{long pyramid}) \leq 3$, or $rw(\text{extended nontrivial basic graph}) \leq 3$.



- Merging two blocks of the 2-join decomposition preserves the rank-width.

8f. Results on width of EHF graphs

Theorem (Adler, et al. (2018))

∃ a family of (even hole, diamond)-free graphs without clique cutsets with unbounded rank-width.

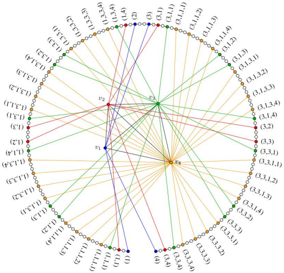


Figure: A diamond-free ehf graph that may have arbitrarily large rank-width

So, excluding clique cutset $\not\Rightarrow$ bounded tree-width.

8g. Tree-width of K_4-free EHF graphs is unbounded

Theorem (S., Trotignon (2019))

There exists a family of (even hole, K_4)-free graphs with arbitrarily large tree-width.

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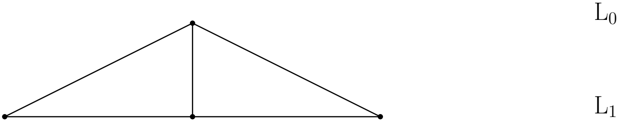

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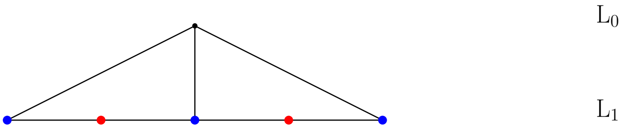

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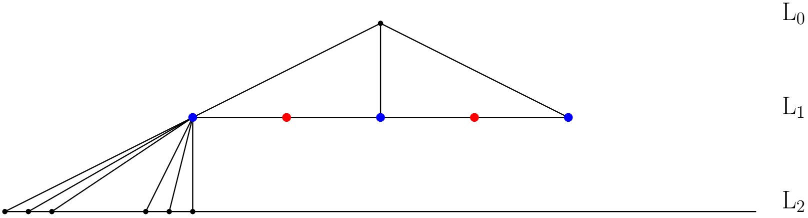

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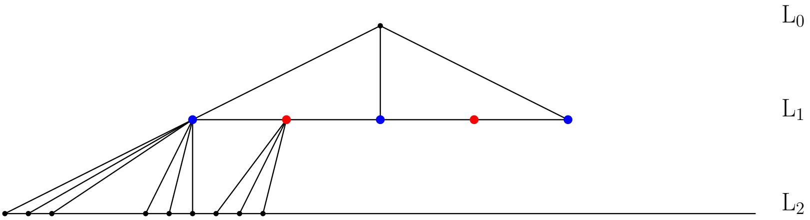

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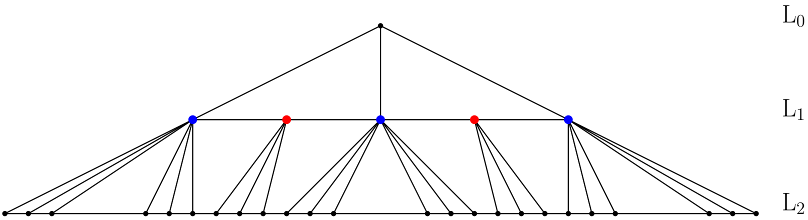

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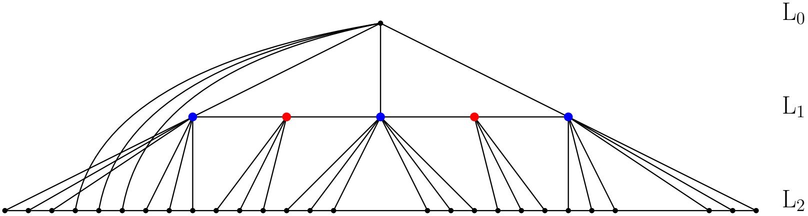

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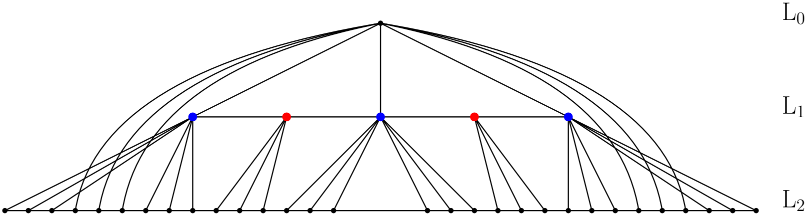

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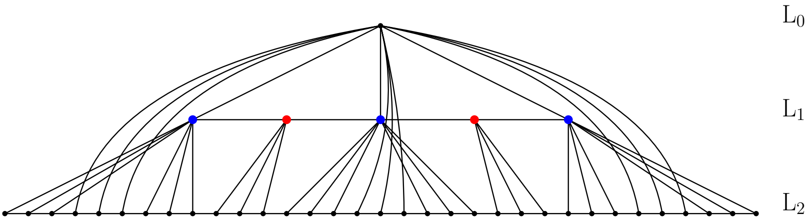

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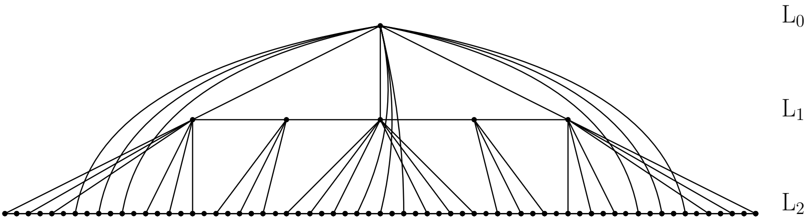

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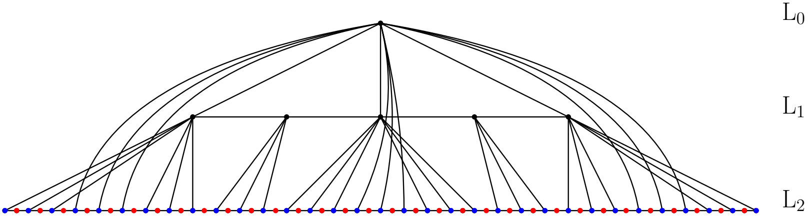
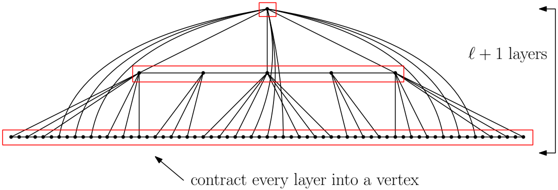


8g. Tree-width of K_4-free EHF graphs is unbounded

Why these graphs have large tree-width?



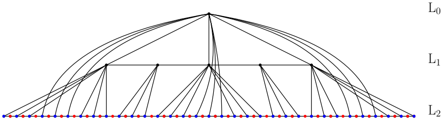
- The graph contains $K_{\ell+1}$-minor.

Theorem (S., Trotignon (2019))

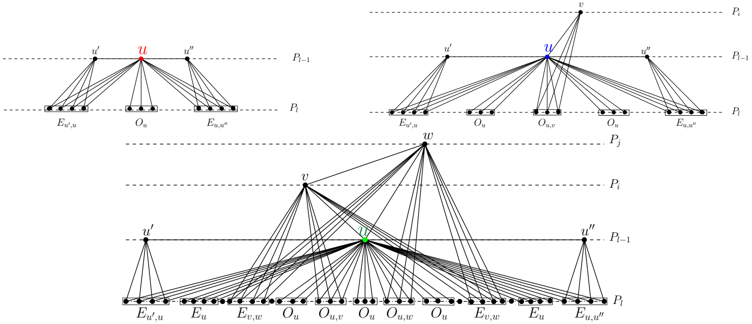
For any $\ell \geq 0$, layered-wheels on $\ell + 1$ layers have tree-width at least ℓ.

8g. Tree-width of K_4-free EHF graphs is unbounded

It was actually cheating... The graph is not an even-hole-free graph



The real construction:



8g. Rank-width of K_4-free EHF graphs is unbounded

Theorem (Gurski, Wanke (1928))

If a graph G contains no $K_{3,3}$ as a subgraph, then $tw(G) \leq 6cw(G) - 1$.

Theorem

A layered wheel contains no $K_{3,3}$ as a subgraph.

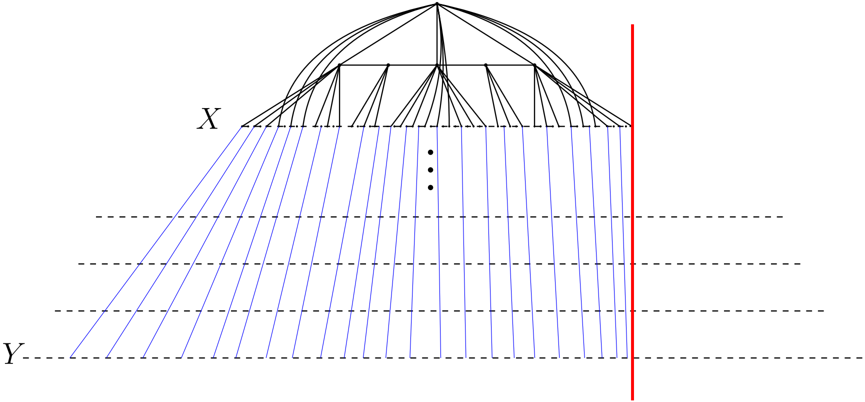
- If one side of the $K_{3,3}$ is a clique, then G contains a K_4.
- Otherwise, each side of $K_{3,3}$ contains a non-edge, so G contains $K_{2,2}$ (i.e. C_4).

Lower bound: the clique-width of a layered wheel on $\ell + 1$ layers is at least $\frac{\ell+1}{6}$.

8g. Rank-width of K_4-free EHF graphs is unbounded

Theorem (S., Trotignon (2019))

For $\ell \geq 2$, there exists a layered wheel G_ℓ with rank-width at least ℓ.



8g. Tree-width of K_4-free EHF graphs is in $\mathcal{O}(\log n)$

Theorem (Cygan, et al. (2015))

$tw(G_\ell) \leq pw(G_\ell) \leq \omega(\mathcal{I}) - 1$, where $\mathcal{I}$ is an interval graph containing G_ℓ as a subgraph

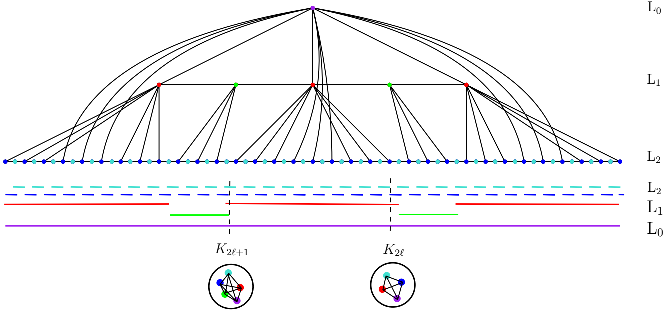


Figure: Interval graph $\mathcal{I}$ that contains a layered wheel with 3 layers

8g. Analysis of layered wheel

- We actually have a parameter k for G_ℓ that determines the girth of G_ℓ. Given large k, this family provides an example of *sparse* graph with high tree-width.
- G_ℓ needs a huge number of vertices to increase the lower bound on the tree-width, and it must contain a vertex of high degree.

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

Theorem (Aboulker, Adler, Kim, S., Trotignon (2021))

An even-hole-free graph G with no K_k-minor satisfies $tw \leq f(k)$.

Induced-wall theorem for H-minor-free graph

Theorem (Aboulker, Adler, Kim, S., Trotignon (2021))

$\forall H$, if G is H-minor-free with $tw(G) \geq f_H(k)$, then G contains a $(k \times k)$-wall (possibly subdivided) or the line graph of a chordless $(k \times k)$-wall (or call it co-wall) as an induced subgraph.

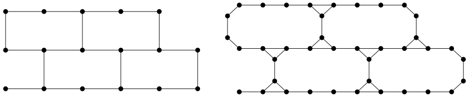


Figure: A $(k \times k)$-wall and $(k \times k)$-co-wall

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

Theorem (Fomin, Golovach, Thilikos, 2011)

For every H and an integer k, there exists a function $f_H(k)$ s.t. for every connected H-minor free graph G with $tw(G) \geq f_H(k)$, G contains either Γ_k or Π_k as a contraction.

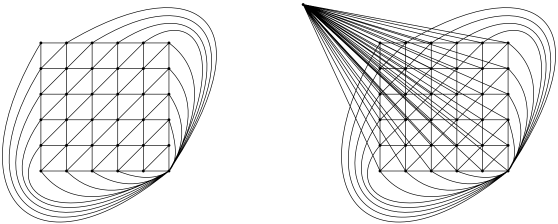


Figure: Γ_6 and Π_6

G' is a contraction of G if G' can be obtained by contracting edges of G

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

Let G be s.t. $tw(G) \geq f_H(k)$, then G contains Γ_k or Π_k

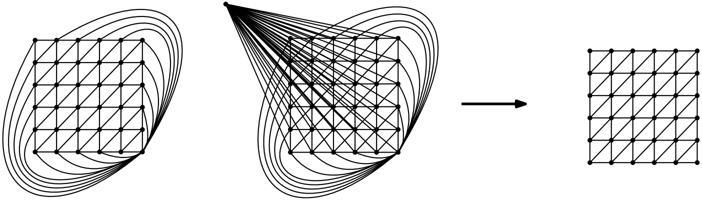


Figure: We can extract a *triangulated grid*

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

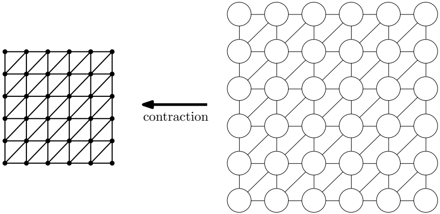


Figure: Consider the graph containing the contracted triangulated grid

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

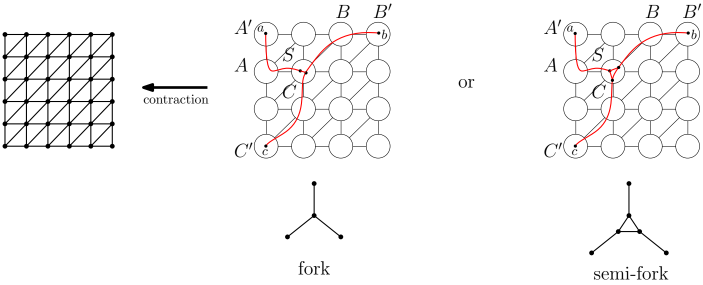


Figure: For some constant size of the triangulated grid, we find *forks* and *semiforks*

8h. EHF graphs with no K_k minor have $tw \leq f(k)$

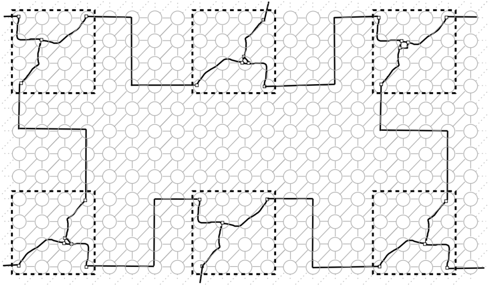


Figure: Combining them, we get a *large stone wall*

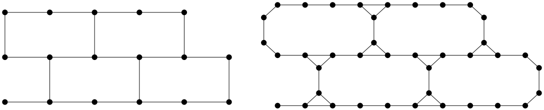


Figure: Applying a "cleaning" procedure, we can extract a $(k \times k)$-wall or a $(k \times k)$-co-wall

8i. EHF graphs with $\Delta \leq 3$ have bounded tree-width

Theorem (Aboulker, Adler, Kim, S., Trotignon (2020))

An even-hole-free graph with maximum degree at most 3 has tree-width at most 3.

Theorem (Aboulker, Adler, Kim, S., Trotignon (2020))

Let G be a (theta, prism)-free subcubic graph. Then either:

- *G is a basic graph; or*
- *G has a clique separator of size at most 2; or*
- *G has a proper separator.*

8i. EHF graphs with $\Delta \leq 3$ have bounded tree-width

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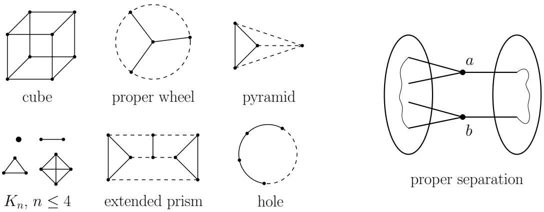


Figure: Basic graphs and proper separator

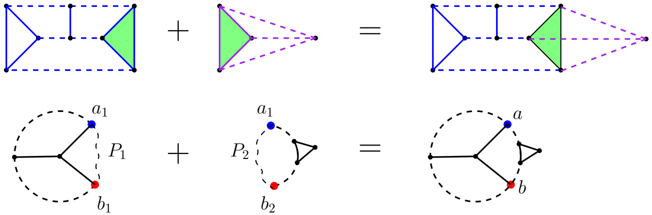
8i. EHF graphs with $\Delta \leq 3$ have bounded tree-width

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

Subcubic even-hole-free graphs have tree-width ≤ 3.

Sketch of proof.

- Every basic graph has tree-width at most 3
- "Gluing" along a clique and proper gluing preserve tree-width



How to prove a class $\mathcal{C}$ has bounded/unbounded $(\cdot)$-width?

- Proving *unbounded tree-width/rank-width* is done by giving a family of graphs in $\mathcal{C}$ whose tree-width grows with the size of the graphs.
- Proving the "bounded" case is done by applying the structural properties of the class (such as: the structure theorem), apply graph chordalization.
- Just a feeling: unboundedness might be "easier" to prove via rank-width than clique-width.